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A Minimax-Measure Intersection Problem

Philip R. Meyers

Institute for Basic Standards, National Bureau of Standards, Washington, D.C. 20234

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The problem solved is that of selecting n subsets of the unit interval, each of measure α, so as to minimize the maximum of the measures of their p-fold intersections. This is achieved by minimizing the sum of the measures of these p-fold intersections.

Key words: Combinatorial analysis; combinatorial probability; measure theory; minimax.

1. Introduction

Some years ago, NBS colleague S. Haber communicated the following problem: To select n subsets of the unit interval, each of measure $1/2$, so as to minimize the maximum of the measures of the pairwise intersections of these subsets. The problem is suggested by a paper [1][1] of Gillis which, settling "an unpublished conjecture of Erdos," proves that for denumerably infinite collections of sets of measure α, the value corresponding to the maximum pairwise-intersection measure has infimum α^2. (Collections with higher transfinite cardinality are treated by Gillis in [2].) Here we provide an explicit solution for collections of *finite* cardinalities n. Further, and also corresponding to [1], we consider as well the case of p-fold intersections with $2 \leq p \leq n$, and provide the corresponding explicit solution. (As noted in [2], the argument of [1] easily extends to show that α^p is the limiting value for a denumerably infinite collection.)

As preliminary, we introduce a second minimization and point out its relationship to our minimax problem, to wit: Select n subsets $A_1, A_2, \ldots, A_n$ of the unit interval, each of measure α, so that the *sum* of the measures of their p-fold intersections is minimum. If now $X = \{S_1, \ldots, S_n\}$, a solution to this minimum problem, can be chosen so that all its p-fold intersections have the *same* measure s, and if M is the maximum of the measures of the p-fold intersections of an arbitrary collection $A_1, A_2, \ldots, A_n$ with all $\mu(A_i) = \alpha$, then

$$\binom{n}{p} M \geq \sum_{i_1<i_2<\cdots<i_p} \mu\left(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_p}\right)$$

$$\geq \sum_{i_1<i_2<\cdots<i_p} \mu\left(S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_p}\right)$$

$$= \binom{n}{p} s.$$

Thus $s \geq M$, demonstrating that X solves the minimax problem. This observation suggested the analysis which follows.

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[1] Figures in brackets indicate the literature references at the end of this paper.

2. Analysis

We will use the following notation: $N = \{1,2,\ldots, n\}$; p is a fixed positive integer with $2 \leq p \leq n$. The underlying space is the unit interval I with Lebesgue measure μ (but the analysis actually carries over to any "atomless" probability space). Set complements of $A \subseteq I$ and $R \subseteq N$ are denoted A^c and R^c respectively. Let, for $0 \leq r \leq n$,

$$K_r = \{R \subseteq N: \ |R| = r\}.$$

Given the real number α with $0 < \alpha \leq 1$, let

$$F(\alpha) = \{A \subseteq I : \mu(A) = \alpha\}$$

and let $F^n(\alpha)$ denote the n-fold Cartesian power of $F(\alpha)$, consisting of all n-tuples

$$X = \{A_1, A_2, \ldots, A_n\}$$

with each $A_i \in F(\alpha)$. For each such X, and each $R \subseteq N$, set

$$X_R = \{x \in I: \ x \in A_i \quad iff\ i \in R\};$$

an easily-proved property of these sets, to be used repeatedly below, is that

$$X_R \cap (\bigcap_{i \in P} A_i) = \begin{cases} X_R \text{ if } P \subseteq R, \\ \phi \text{ otherwise.} \end{cases} \tag{1}$$

Note that the disjoint union

$$X_r = \bigcup_{R \in K_r} X_R$$

consists of those points $x \in I$ which are members of exactly r sets $A_i \in X$. Finally, for measurable B

$\subseteq I$, it is convenient to define

$$S(X,B) = \sum_{P \epsilon K_p} \mu \left[\left(\bigcap_{i \epsilon P} A_i \right) \cap B \right].$$

The "objective function" for the minimax problem is

$$M(X) = \max_{P \epsilon K_p} \mu \left[\bigcap_{i \epsilon P} A_i \right],$$

while that for the related minimization problem introduced in section 1 is

$$S(X) = S(X,I) = \sum_{P \epsilon K_p} \mu \left[\bigcap_{i \epsilon P} A_i \right].$$

An alternative formula for $S(X)$ will first be developed (Lemma 1), and then a necessary condition (Lemma 2) for some $X \epsilon F^n(\alpha)$ to minimize S will be presented.

LEMMA 1: *For each* $X \epsilon F^n(\alpha)$,

$$S(X) = \sum_{r=p}^{n} \binom{r}{p} \mu \left(X_r \right). \tag{2}$$

PROOF: Since $\{X_r\colon \ r = 0, 1, \ldots, n\}$ is a partition of I,

$$\begin{aligned} S(X) &= \sum_{r=0}^{n} S(X,X_r) \\ &= \sum_{r=0}^{n} \sum_{|R|=r} S(X,X_R). \end{aligned}$$

Applying (1) to each summand, we obtain

$$S(X) = \sum_{r=p}^{n} \sum_{|R|=r} \binom{r}{p} \mu \left(X_R \right),$$

yielding (2).

LEMMA 2. *If* X *minimizes* S *over* $F^n(\alpha)$, *and* $\mu(X_r) > 0$ *for some* $r \geq p$, *then* $\mu(X_t) = 0$ *for all* $t < r - 1$.

PROOF: Suppose, to the contrary, that there exist $r \geq p$ and $t < r - 1$ such that $\mu(X_r) > 0$ and $\mu(X_t) > 0$. We will prove the existence of an $X' \in F^n(\alpha)$ for which $S(X') < S(X)$, thus contradicting the hypothesis about X.

Since $\mu(X_r) > 0$ and $\mu(X_t) > 0$, K_r and K_t must contain respective members R and T with $\mu(X_R) > 0$ and $\mu(X_T) > 0$. Choose subsets Y and Z of I with

$$Y \subseteq X_R, \qquad Z \subseteq X_T, \qquad \mu(Y) = \mu(Z) > 0.$$

Also choose a member i of the nonempty set $R - T$; then

$$Y \subseteq A_i; \qquad Z \subseteq A_i^c.$$

Now define $X' = \{A_1, A_2, \ldots, A_i', \ldots, A_n\}$, where

$$A_i' = (A_i - Y) \cup Z;$$

Since $\mu(A_i') = \mu(A_i - Y) + \mu(Z) = \mu(A_i)$, we have $X' \in F^n(\alpha)$.

To prove that $S(X)') < S(X)$, observe that I is partitioned into Y, Z, and $I - Y - Z$.

Thus

$$S(X) = S(X,Y) + S(X,Z) + S(X,I - Y - Z),$$

$$S(X') = S(X',Y) + S(X',Z) + S(X',I - Y - Z).$$

Since X and X' differ only on $Y \cup Z$, it follows that

$$S(X) - S(X') = [S(X,Y) - S(X',Y)] - [S(X',Z) - S(X,Z)].$$

Since $Y \subseteq X_r$ and $Y \subseteq X'_{r-1}$, application of (1) to the summands of $S(X,Y)$ and $S(X',Y)$ yields

$$S(X,Y) - S(X',Y) = \binom{r}{p}\mu(Y) - \binom{r-1}{p}\mu(Y) = \binom{r-1}{p-1}\mu(Y).$$

Similarly, it follows from $Z \subseteq X_t$ and $Z \subseteq X'_{t+1}$ that

$$S(X',Z) - S(X,Z) = \binom{t+1}{p}\mu(Z) - \binom{t}{p}\mu(Z) = \binom{t}{p-1}\mu(Z).$$

Since $r - 1 > t$ and $\mu(Y) = \mu(Z) > 0$,

$$S(X) - S(X') = \binom{r-1}{p-1}\mu(Y) - \binom{t}{p-1}\mu(Z) > 0,$$

completing the proof.

We will subsequently show that if $\mu(X_r) > 0$ for some $r > p$ then $\mu\,(X_t) = 0$ for $t < r - 1$ is a sufficient condition for X to minimize S over $F^n(\alpha)$.

Lemma 3: *For all* X ϵ $F^n(\alpha)$,

$$n\alpha = \textstyle\sum_{r=0}^{n} \mu(X_r). \tag{3}$$

Proof: Let c_i denote the characteristic function of A_i. Then

$$n\alpha = \textstyle\sum_{i=1}^{n} \displaystyle\int_I c_i(x)\, d\mu(x) = \int_I [\textstyle\sum_{i=1}^{n} c_i(x)]\, d\mu(x)$$

$$= \textstyle\sum_{r=0}^{n} \displaystyle\int_{X_r} [\textstyle\sum_{i=1}^{n} c_i(x)]\, d\mu(x) = \textstyle\sum_{r=0}^{n} r\mu(X_r).$$

It is now possible to prove:

Lemma 4: If $n\alpha \leq p - 1$, then $S_{\min} = \min\,\{S(Y){:}\;\; Y\;\epsilon F^n(\alpha)\} = 0$.

Proof: It suffices to exhibit an $X\;\epsilon\; F^n(\alpha)$ for which

$$\mu(X_r) = 0 \qquad \text{for } r \geq p. \tag{4}$$

To this end, let

$$A_i = [(i - 1)\alpha,\, i\alpha) \qquad (\text{mod } 1) \qquad \text{for } 1 \leq i \leq n.$$

Each point of [0,1) corresponds (mod 1) to exactly $p - 1$ points of the interval $[0,\, p - 1)$, and thus to at most $p - 1$ points of the subinterval $[0,\, n\alpha)$; thus $X_r \cap [0,\,1) = \phi$ for $r \geq p$, verifying (4).

Lemma 5: *If for given* X ϵ $F^n(\alpha)$, *the largest* r *such that* $\mu(X_r) > 0$ *satisfies* $r \geq p$ *and further for* $t < r - 1$, $\mu(x_t) = 0$, *then*

$$S(X) = S_{\min}.$$

Proof. It suffices to show that $S(X)$ has the same value for all $X\;\epsilon\; F^n(\alpha)$ satisfying the conditions of the lemma. Consider such an X, and the greatest r for which $\mu(X_r) > 0$. Since

$$\sum_{t=0}^{n} \mu(X_t) = \mu(I) = 1, \tag{5}$$

such an r must exist. By the above condition $\mu(X_t) = 0$ for $t \neq r,\, r - 1$, and so by (5),

$$\mu(X_{r-1}) = 1 - \mu(X_r).$$

Let $n\alpha = m + \beta$ with m integral and $0 \leq \beta < 1$. It follows from (3) that

$$m + \beta = r\mu(X_r) + (r - 1)\,\mu(X_{r-1}) = (r - 1) + \mu(X_r), \tag{6}$$

and from (2) that

$$S(X) = \binom{r}{p}\mu(X_r) + \binom{r-1}{p}\mu\left(X_{r-1}\right) = \binom{r-1}{p} + \binom{r-1}{p-1}\mu(X_r). \tag{7}$$

If $\beta = 0$, then since m is integral and $0 < \mu(X_r) \leq 1$, it follows from (6) that $\mu(X_r) = 1$ and $r = m$, and then it follows from (7) that

$$S(X) = \binom{m-1}{p} + \binom{m-1}{p-1} = \binom{m}{p}. \tag{8}$$

If $\beta > 0$, then it follows from (6) that $\mu(X_r) = \beta$ and $m = r - 1$, and then it follows from (7) that

$$S(X) = \binom{m}{p} + \binom{m}{p-1}\beta. \tag{9}$$

Thus $S(X)$ is uniquely determined by the pair (m, β), i.e., by $n\alpha$. Note that (8) and (9) are consistent with Lemma 4, since both yield $S(X) = 0$ if $n\alpha \leq p - 1$.

We are now able to provide the solutions, both to the problem of minimizing $S(X)$ over $F^n(\alpha)$ and to the original problem of minimizing

$$M(X) = \max_{P \epsilon K_p} \mu\left(\bigcap_{i \epsilon P} A_i\right)$$

over $F^n(\alpha)$. Let $M_{\min}$ denote the value of this latter minimum. Then the solution takes the following form.

THEOREM. *Let* $n\alpha = m + \beta$ *with* m *integral and* $0 \leq \beta < 1$. *Then*

$$S_{\min} = M_{\min} = 0 \qquad \textit{if } n\alpha \leq p - 1,$$

$$S_{\min} = \binom{m}{p} + \binom{m}{p-1}\beta, \qquad M_{\min} = S_{\min}\Big/\binom{n}{p} \qquad \textit{if } n\alpha > p - 1.$$

Thus, in particular, for the problem as originally posed where p=2 *and* $\alpha = 1/2$,

$$S_{\min} = \begin{cases} k(k - 1)/2 & \text{if } n=2k \\ (k - 1)^2/2 & \text{if } n=2k - 1, \end{cases}$$

and $M_{\min} = (k - 1)/2(2k - 1)$.

PROOF: First suppose $n\alpha \leq p - 1$. Then $S_{\min} = 0$ follows from Lemma 4, whose proof constructed an $X \epsilon F^n(\alpha)$ for which $\mu\,(\cup_{r=p}^{n} X_r) = 0$. Since every p-fold intersection of the members of X lies in this union, it follows that $M(X) = 0$, implying $M_{\min} = 0$.

Now suppose $n\alpha > p - 1$. The formula for $S_{\min}$ follows from (8) and (9). We will prove the result for $M_{\min}$ by constructing an $X \in F^n(\alpha)$ which satisfies the condition of Lemma 2, and which furthermore (see the end of sec. 1) has equal measures for each of its p-fold intersections. For this purpose, partition the interval $[0, \beta)$ into $\binom{n}{m+1}$ equal subintervals and the interval $[\beta, 1]$ into $\binom{n}{m}$ equal subintervals. Label the second family of subintervals as $\{X_M: \ M \in K_m\}$ and the first family as $\{X_Q: \ Q \in K_{m+1}\}$. Define

$$A_i = [\cup\{X_M: \ i \in M\}] \cup [\cup\{X_Q: \ i \in Q\}].$$

Then each A_i consists of $\binom{n-1}{m-1}$ intervals X_M and $\binom{n-1}{m}$ intervals X_Q, all disjoint, so all A_i have equal measure. If c_i denotes the characteristic function of A_i, then

$$\sum_{i=1}^{n} \mu(A_i) = \int_I \left(\sum_{i=1}^{n} c_i\right) d\mu = \int_0^\beta \left(\sum_{i=1}^{n} c_i\right) d\mu + \int_\beta^1 \left(\sum_{i=1}^{n} c_i\right) d\mu$$

$$= (m+1)\beta + m(1-\beta) = m + \beta = n\alpha.$$

Thus each $\mu(A_i) = \alpha$, i.e., $X \in F^n(\alpha)$. For $r \geq p$, $\mu(X_r) > 0$ holds only for $r = m$ and $r = m + 1$, so the condition of Lemma 2 is satisfied. The symmetry of the construction assures that all p-fold intersections of the members of X have equal measure; explicitly, for $P \in K_p$, we have

$$\bigcap_{i \in P} A_i = [\cup\{X_M: \ P \subset M\}] \cup [\cup\{X_Q: \ P \subset Q\}].$$

implying

$$\mu[\bigcap_{i \in P} A_i] = \binom{n-p}{m-p}(1-\beta)/\binom{n}{m} + \binom{n-p}{m+1-p}\beta/\binom{n}{m+1},$$

independently of P.

3. References

[1] Gillis, J., Note on a property of measurable sets, J. London Math. Soc. **11**, 139–141 (1936).

[2] Gillis, J., Some combinatorial properties of measurable sets, Quart. J. Math. (Oxford) **7**, 191–198 (1936).

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